12th Edition

A Problem Solving Approach to Mathematics

for Elementary School Teachers

Billstein

Libeskind

Lott

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Dedication

To my students who for the past 46 years have provided me with enjoyment and challenge.
Each new edition of this book reflects experiences learned in the classroom.
—Rick Billstein

To the American troops in Munich, Germany (1945–1948), who offered hope and
protection to survivors of the Holocaust.
—Shlomo Libeskind

To Carolyn for her support in all endeavors for many years, and to the next generation
of prospective mathematics teachers without whom both students
and mathematics would be in serious trouble.
—Johnny W. Lott

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Contents

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Preface

The twelfth edition of *A Problem Solving Approach to Mathematics for Elementary School Teachers* is designed to prepare outstanding future elementary and middle school teachers. This edition continues to be heavily concept- and skill-based, with an emphasis on active and collaborative learning. The content has been revised and updated to better prepare students to become teachers in their own classrooms.

National Standards for Mathematics

- **Common Core State Standards for Mathematics** The National Governors Association spearheaded the effort to develop the *Common Core Standards (2010)*; they have been adopted by many states and are used in this text to highlight concepts. The complete text of the *Common Core Standards* is found at www.corestandards.org.
- **Principles and Standards** The National Council of Teachers of Mathematics (NCTM) publication, *Principles and Standards of School Mathematics* (2000) continues to be a guide for the course. The complete text of the NCTM *Principles and Standards* can be found online at www.nctm.org.

Our Goals

- To present appropriate mathematics in an intellectually honest and mathematically correct manner.
- To use problem solving as an integral part of mathematics.
- To approach mathematics in a sequence that instills confidence and challenges students.
- To provide opportunities for alternate forms of teaching and learning.
- To provide communication and technology problems to develop writing skills that allow students to practice reasoning and explanation through mathematical exposition.
- To provide core mathematics for prospective elementary and middle school teachers in a way that challenges them to determine why mathematics is done as it is.
- To provide core mathematics that allows instructors to use methods integrated with content.
- To assist prospective teachers with connecting mathematics, its ideas, and its applications.
- To assist future teachers in becoming familiar with the content and philosophy of the national standards listed above.

The twelfth edition gives instructors a variety of approaches to teaching, encourages discussion and collaboration among future teachers and with their instructors, and aids the integration of projects into the curriculum. Most importantly, it promotes discovery and active learning.

New to This Edition

- At reviewers' suggestions, we **moved topics related to logic** from Chapter 1 to Chapter 2, where sets and the operations of union and intersection are covered.
- **Learning Objectives** are listed at the beginning of every section to focus student attention on the key ideas.
- This text has always reflected the content and processes set forth in today's new state mathematics standards and the Common Core State Standards (CCSS). In the twelfth edition, we have further **tightened the connections to the standards** and made them more explicit in the narrative and exercises:
  - **CCSS are cited within sections** to focus student attention and provide a springboard for discussion of their content.
  - **More exercises that address the CCSS** have been added, particularly in the Mathematical Connections portion of the exercise sets.
  - The **treatment of many topics has been enhanced** to reflect a tighter connection to the CCSS. Examples include:
    - Chapter 1: Expanded the Four-Step Polya Problem solving process with input from Standards for Mathematical Practice. The process is referred to in examples throughout the chapter.
    - Chapter 2: Moved the logic section from Chapter 1 to emphasize the connections to sets and language. Logical reasoning is now an integral part of Chapter 2.
    - Chapter 5: Now includes a definition of addition for integers that uses absolute value—included because it is one of the techniques used in operations on integers in CCSS.

- Chapter 6: The section on ratio and proportion now uses the types of diagrams to set up the proportions that are mentioned in CCSS.
- Chapter 8: Algebraic Thinking is extended to real numbers with greater emphasis on multistep word problems, as described in CCSS.
- Chapter 13: Following CCSS emphasis on transformations and symmetry, these topics are expanded in exposition and in problem sets. New engaging problems were added.
- Chapter 14: All measurement topics are now together in this chapter. Linear measure had been separated out, but because of measurement being highlighted in CCSS, all of the topics are in the same chapter.

- The **text has been streamlined** to help students focus on what's really needed. We made judicious cuts with the student in mind.
- Some of the chapter opener scenarios and exercises have been revised to make them more relevant and engaging.
- The **chapter summary charts have been revised** to make them more comprehensive resources for students as they prepare for tests.

Content Highlights

Chapter 1 An Introduction to Problem Solving
This chapter has been reorganized and shortened to make it friendlier. Much of the detail work on series has been moved to later chapters to allow students to gain a knowledge of problem-solving techniques with less algebraic manipulation at this stage.

Chapter 2 Introduction to Logic and Sets
This chapter has been reorganized to include a section on logic. It works hand in hand with the ideas of set operations and enhances reasoning. Set theory and set operations with properties are introduced as a basis for learning whole number concepts.

Chapter 3 Numeration Systems and Whole Number Operations
This chapter models addition and subtraction of whole numbers. It emphasizes the *missing-addend model*, the definition of subtraction in terms of addition, and discusses various algorithms for addition and subtraction including those in different bases. Models for multiplication and division of whole numbers, properties of these operations with emphasis on the distributive property of multiplication over addition, and various algorithms are covered in depth. Mental mathematics and estimation with whole numbers feature prominently.

Chapter 4 Number Theory
In the twelfth edition, a separate chapter on number theory does not depend on *integers*, which are introduced in Chapter 5. Concepts of divisibility with divisibility tests are discovered. Prime numbers, prime factorization, greatest common divisor and least common multiple as well as the Euclidean Algorithm are explored with many new exercises added. A module on Clock Arithmetic is available online.*

Chapter 5 Integers
This chapter concentrates only on integers, their operations, and properties.

Chapter 6 Rational Numbers and Proportional Reasoning
This chapter has been revised to follow many recommendations in the *Common Core Standards*. Videos showing elementary students learning fraction concepts are included so that future teachers can observe what happens when elementary students absorb what is taught and how they work with those concepts. Proportional reasoning, one of the most important concepts taught in middle school mathematics, is covered in great depth in its natural setting.

Chapter 7 Rational Numbers as Decimals and Percent
This chapter focuses on decimal representation of rational numbers. Discussion of percent includes the computing of simple and compound interest as well as estimation involving percents.

Chapter 8 Real Numbers and Algebraic Thinking
With an introduction to real numbers in the opening sections, the chapter combines knowledge of real numbers with algebraic skills to give a review of algebra needed to teach in grades K through 8. This includes work in the coordinate plane and with spreadsheets. A module on Using Real Numbers in Equations is available online.*

*Online modules are availale in MyMathLab or at www.pearsonhighered.com/mathstatsresources

Chapter 9 Probability
This chapter has been reorganized with odds now as an application of probability. *Common Core Standards* have been addressed with content designed to accompany these standards.

Chapter 10 Data Analysis/Statistics: An Introduction
Chapter 10 opens with Designing Experiments/Collecting Data, a section based on *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report: A preK–12 Curriculum Framework* (2005) by the American Statistical Association. This section, aligned with the *Common Core Standards*, focuses on designing studies and surveys. In the following sections, data, graphs, examples, and assessment exercises have been updated and new material added.

Chapter 11 Introductory Geometry
This chapter has been reorganized to allow students to explore some of the ramifications of different definitions in mathematics used in schools. Linear measure is introduced to emphasize its importance in the curriculum. Also symmetries are now introduced as an early concept that could be used to form geometrical definitions. The Networks module is now offered online.*

Chapter 12 Congruence and Similarity with Constructions
Congruence and constructions sections have been expanded to allow more exploration of circles and quadrilaterals. The concept of similarity is used to reintroduce slope of a line and its properties. Many new exercises have been added. A module on Trigonometric Ratios via Similarity is available online.*

Chapter 13 Congruence and Similarity with Transformations
Because of the prominence of motion geometry in the *Common Core Standards*, this chapter appears earlier among the geometry sections. It focuses on connections among transformations and dilations in congruence and similarity.

Chapter 14 Area, Pythagorean Theorem, and Volume
Chapter 14 continues a reorganization of the geometry chapters. Concepts of linear measure is included with the topics of area, the Pythagorean theorem, and volume. Many topics have been shifted and new material added, for example, the subsection *Comparing Volumes of Similar Figures*. Assessment sets and examples have been updated.

Technology Usage
Virtually all mathematics standards have included the use of technology as a tool for learning mathematics, yet the manner and type of usage in classrooms is as varied as the classrooms and teachers themselves. We strongly support the use of technology as a learning tool and have since the inception of this book. In this edition, online modules discuss the use of technology*. These modules are designed for a brief introduction to the use of spreadsheets and graphing calculators as indicated but it is expected that many instructors using the text will naturally incorporate those tools in their teaching. Additionally, a module on the use of *GeoGebra* is available.

References to the online geometry module problems and lab activities are included in the Mathematical Connections section of the assessments under the heading GeoGebra Activities. It is noted that there are more problems and activities in the online modules than are listed in the text. This is purposefully done to allow instructors to use them in the manner that is most pedagogically and mathematically desirable for their courses.

Features

In creating the 12th edition of this text, we have built upon the strengths of the previous editions, incorporating feedback from users and making extensive improvements to help prepare future teachers for new state standards and the Common Core.

Learning the Mathematics in the New Standards

- **New!** In this edition we have made judicious cuts to even more effectively bring key ideas to the forefront. **A streamlined narrative** keeps students focused on the important ideas.
- **Preliminary Problems** open every chapter with a thought-provoking question that sets the tone and prepares students for the material ahead.
- **New! Learning Objectives** are listed at the beginning of every section to focus student attention on the key ideas.

*Online modules are available in MyMathLab or at www.pearsonhighered.com/mathstatsresources

- **Problem-Solving Strategies** are highlighted in italics, and **Problem-Solving Boxes** throughout the text help students put these strategies to work.
- **Chapter Summaries** are organized in a student-friendly chart format, for easy exam preparation. These summaries also provide specific pages references for additional help on individual concepts.
- **Chapter Review** questions allow students to test themselves when preparing for an exam.

Focusing on the Standards for Mathematical Practice

- The **Activity Manual** includes classroom-tested activities and a pouch of perforated, printed color manipulatives.
  - **Activity Manual annotations** in the Annotated Instructor's Edition clarify when specific activities should be used for each lesson, making it easier to teach a more hands-on course.
  - The manual is available as a value-pack option. Ask your Pearson representative for details.
- **Now Try This** exercises, which follow key examples, help students become actively involved in their learning, facilitate the development of critical thinking and problem-solving skills, and stimulate class discussion. Answers are in both the Annotated Instructor's Edition and student text.
- **21 eManipulatives,** available in MyMathLab, allow students to investigate, explore, practice, build conceptual understanding, and solve specific problems, without the mess or cost of physical manipulatives. Annotations in the student edition indicate where these eManipulatives are relevant. Exercises related to the eManipulatives are assignable within MyMathLab.
- **Integrating Mathematics and Pedagogy (IMAP)** videos, available in MyMathLab, feature elementary school children working problems. Annotations in the student edition indicate where these videos are relevant. Exercises related to the IMAP videos are assignable within MyMathLab.

Teaching the Mathematics in the New Standards

- **New! Common Core State Standards (CCSS)** are cited within sections to focus student attention and provide a springboard for discussion of their content.
- **New! More exercises** have been added that address the CCSS, particularly in the Mathematical Connections portions of the exercise sets.
- **Connecting Mathematics to the Classroom** exercises require interpretation and analysis of the thinking of typical K-8 students.
- **School book pages** are included to show how various topics are introduced to the K-8 pupil. Icons within the text link the narrative to the appropriate school book page. Students are asked to complete many of the activities on the student pages so they can see what is expected in elementary school.
- **Historical Notes** add context and humanize the mathematics.
- **New!** Enhanced **Common Core State Standards (CCSS)** coverage in MyMathLab encourages students to become familiar with important content and procedures. The **view by standard** functionality in MyMathLab also includes CCSS.

Assessing the New Standards

- **Extensive Problem Sets** are organized into three categories for maximum instructor flexibility when assigning homework that address the standards.
  - **Assessment A** has problems with answers in the text, so that students can check their work.
  - **Assessment B** contains parallel problems to those in Assessment A, but answers are not given in the student text.
  - **Mathematical Connections** problems include the following categories: Reasoning, Open-Ended, Cooperative Learning, Connecting Mathematics to the Classroom, Review Problems, and NAEP sample questions.
- **Hundreds of assignable, algorithmic exercises.** The MyMathLab courses for the Twelfth Edition contains even more assignable exercises to meet students' needs. Assignable exercise types include the following:
  - **Textbook exercises**—over 2,000 algorithmically generated exercises parallel those in the text
  - **New! Common Core Assessment Analysis** exercises require analysis and interpretation of sample CCSS exercises.
  - **eManipulative exercises** require use of the eManipulatives within MyMathLab so students can be familiar with this important teaching and learning tool.
  - **Integrating Mathematics and Pedagogy (IMAP)** video exercises require analysis of student work.
  - **Assessment exercises** include hundreds of exercises from the test bank.

Student and Instructor Resources

For the Student

Activities Manual
***Mathematics Activities for Elementary School Teachers: A Problem Solving Approach*, 12th edition**
Dan Dolan, *Project to Increase Mastery of Mathematics and Science*, *Wesleyan University*; Jim Williamson, *University of Montana*; and Mari Muri, *Project to Increase Mastery of Mathematics and Science*, *Wesleyan University*
ISBN 0-321-97708-4 | 978-0-321-97708-3

- Provides hands-on, manipulative-based activities keyed to the text that involve future elementary school teachers discovering concepts, solving problems, and exploring mathematical ideas.
- Colorful, perforated paper manipulatives in a convenient storage pouch.
- Activities can also be adapted for use with elementary students at a later time.
- References to these activities are in the margin of the Annotated Instructor's Edition.

Student's Solutions Manual
Barbara Boschmans, *Northern Arizona University* and Brian Beaudrie, *Northern Arizona University*
ISBN 0-321-99056-0 | 978-0321-99056-3

- Provides detailed, worked-out solutions to all of the problems in **Assessment A**, odd **Mathematical Connections Review** problems, and all **Chapter Review** exercises.

For the Instructor

Annotated Instructor's Edition
ISBN 0-321-99044-7 | 978-0-321-99044-0

- This special edition includes answers to the text exercises on the page where they occur and includes answers to the Preliminary Problems, Now Try This activities, and Mathematical Connections questions.
- Annotations referencing the *Activities Manual* are included in the margins.

Online Supplements

The following instructor material is available for download from Pearson's Instructor Resource Center (www.pearsonhighered.com/irc) or within MyMathLab.

Instructor's Solutions Manual
Barbara Boschmans, *Northern Arizona University* and Brian Beaudrie, *Northern Arizona University*

- Provides detailed, worked-out solutions to all of the problems in **Assessments A and B**, **Mathematical Connections Review** problems, and **Chapter Review** exercises.

Instructor's Testing Manual

- Comprehensive worksheets contain two forms of chapter assessments with answers for each.

Instructor's Guide for
***Mathematics Activities for Elementary School Teachers: A Problem Solving Approach*, 12th edition**
Dan Dolan, *Project to Increase Mastery of Mathematics and Science*, *Wesleyan University*; Jim Williamson, *University of Montana*; and Mari Muri, *Project to Increase Mastery of Mathematics and Science*, *Wesleyan University*

- Contains answers for all activities, as well as additional teaching suggestions for some activities.

PowerPoint® Lecture Slides

- Fully editable slides provide section-by-section coverage of key topics and concepts along with examples.
- An Image Resource Library is also available within MyMathLab and contains art files from the text enabling further customization of the PowerPoint lectures.

TestGen®

- TestGen® enables instructors to build, edit, print, and administer tests using a computerized bank of questions developed to cover all the objectives of the text.
- TestGen is algorithmically based, allowing instructors to create multiple but equivalent versions of the same question or test with the click of a button. Instructors can also modify test bank questions or add new questions.

Online Learning

MyMathLab® Online Course (access code required)

MyMathLab from Pearson is the world's leading online resource in mathematics, integrating interactive homework, assessment, and media in a flexible, easy to use format.

MyMathLab delivers **proven results** in helping individual students succeed.

- MyMathLab has a consistently positive impact on student retention, subsequent success, and overall achievement. MyMathLab can be successfully implemented in any environment—lab-based, hybrid, fully online, or traditional.
- MyMathLab has a comprehensive online gradebook that automatically tracks your students' results on tests, quizzes, homework, and in the study plan. You can use the gradebook to quickly intervene if your students have trouble, or to provide positive feedback on a job well done.

MyMathLab provides **engaging experiences** that personalize, stimulate, and measure learning for each student.

- **Personalized Learning:** MyMathLab's personalized homework and adaptive study features allow your students to work more efficiently, spending time where they really need to.
- **Exercises:** The homework and practice exercises in MyMathLab are correlated to the exercises in the textbook, and they regenerate algorithmically to give students unlimited opportunity for practice and mastery. The software offers immediate, helpful feedback when students enter incorrect answers. Assignable exercise types include the following:
  - **Textbook exercises**—over 2,000 algorithmically generated exercises parallel those in the text
  - **New! Common Core Assessment Analysis** exercises in MyMathLab require analysis and interpretation of sample CCSS exercises.
  - **eManipulatives exercises** require use of the eManipulatives within MyMathLab so students can be familiar with this important teaching and learning tool.
  - **Integrating Mathematics and Pedagogy (IMAP)** video exercises require analysis of student work.
  - **Assessment exercises** include hundreds of exercises from the test bank.
- **Multimedia Learning Aids:** Exercises include guided solutions, sample problems, videos, and access to the complete eText access.
- **Complete eText** is available to students for the life of the edition, giving students unlimited access to the eBook within any course that uses that edition of the textbook.
- **eManipulatives** allow students to investigate, explore, practice, build conceptual understanding, and solve specific problems, without the mess or cost of physical manipulatives.
- **New!** Enhanced **Common Core State Standards (CCSS)** coverage in MyMathLab encourages students to become familiar with important content and procedures. The **view by standard** functionality in MyMathLab also includes CCSS.
- **New! Study Skills resources** help students develop good time management skills and deal with stress management.
- **A wide array of videos** meets the unique needs of future teachers.
  - **Integrating Mathematics and Pedagogy (IMAP)** videos feature elementary school children working problems. Exercises related to these videos are assignable within MyMathLab.
  - **New! Section Lecture** videos are revised for this edition—ideal for studying and reviewing.
  - **New! Common Core in Action videos** featuring experienced faculty shedding light on what the CCSS really mean for the classroom and for teachers.
  - **Classroom** videos show experienced teachers teaching key topics, with commentary by the teacher and college faculty.
  - **Responding to Student Work** videos contain analysis and helpful responses to elementary school children's work.
- **Accessibility:** MyMathLab is compatible with the JAWS screen reader and enables multiple-choice and free-response problem types to be read and interacted with via keyboard controls and math notation input. MyMathLab also works with screen enlargers, including ZoomText, MAGic, and SuperNova. And all MyMathLab videos in this course are closed captioned. More information on this functionality is available at http://mymathlab.com/accessibility.

And, MyMathLab comes from an experienced partner with educational expertise and an eye on the future.

- Whether you are just getting started with MyMathLab, or have a question along the way, we're here to help.
- Contact your Pearson representative directly or at www.mymathlab.com.

MyMathLab® Ready to Go Course (access code required)

These new Ready to Go courses provide students with all the same great MyMathLab features that you're used to, but make it easier for instructors to get started. Each course includes pre-assigned homeworks and quizzes to make creating your course even simpler. Ask your Pearson representative about the details for this particular course or to see a copy of this course.

MathXL® Online Course (access code required)

MathXL® is the homework and assessment engine that runs MyMathLab. (MyMathLab is MathXL plus a learning management system.) With MathXL, instructors can:

- Create, edit, and assign online homework and tests using algorithmically generated exercises correlated at the objective level to the textbook.
- Create and assign their own online exercises.
- Maintain records of all student work tracked in MathXL's online gradebook.

With MathXL, students can:

- Take chapter tests in MathXL and receive personalized study plans and/or personalized homework assignments based on their test results.
- Use the study plan and/or the homework to link directly to tutorial exercises for the objectives they need to study.
- Access supplemental animations and video clips directly from selected exercises.

MathXL is available to qualified adopters. For more information, visit the website at www.mathxl.com or contact a Pearson representative.

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Acknowledgments

For past editions of this book, many noted and illustrious mathematics educators and mathematicians have served as reviewers. To honor the work of the past as well as to honor the reviewers of this edition, we list all but place asterisks by this edition's reviewers.

Leon J. Ablon
Paul Ache
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Haldon Anderson
Bernadette Antkoviak
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Jim Williamson
Ken Yoder
Jerry L. Young
Deborah Zopf

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BREAKTHROUGH

To improving results

The Best Preparation for New State Standards & Assessments

4-2 Prime and Composite Numbers

4-2 Objectives

Students will be able to understand and explain

- Prime and composite numbers.
- The number of divisors of any whole number.
- The Fundamental Theorem of Arithmetic.
- The factorization of whole numbers.

CCSS The grade 4 *Common Core Standards* states that students should be able to:

Find all factor pairs for a whole number in the range 1–100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1–100 is prime or composite. (p. 29)

One method used in elementary schools to determine the positive factors of a nonzero whole number is to use squares of paper or cubes to represent the number as a rectangle. Such a rectangle resembles a candy bar formed with small squares. The dimensions of the rectangle are divisors or factors of the number. For example, Figure 5 shows rectangles to represent 12.

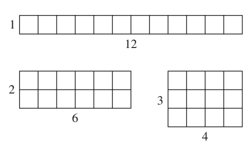


Figure 5

NEW! Relevant excerpts from the Common Core State Standards are incorporated throughout the text so that students see how the standards relate to what they are learning.

NEW! Learning objectives are provided for every section to focus student attention on the key ideas.

Extensive problem sets contain many types of exercises that reflect the rigor of the new CCSS assessments.

Each problem set has three parts:

1. **Assessment A** – focused on skills from the section
2. **Assessment B** – parallels Assessment A, with no answers provided in the student text
3. **Mathematical Connections** – rich conceptual exercises of a variety of types that require students to communicate mathematically

Mathematical Connections 4-3

Reasoning

1. Can two nonzero whole numbers have a greatest common multiple? Explain your answer.
2. Is it true that $\text{GCD}(a, b, c) \cdot \text{LCM}(a, b, c) = abc$? Explain your answer.
3. Suppose that $\text{GCD}(a, b, c) = 1$. Is it necessarily true that $\text{GCD}(a, b) = \text{GCD}(b, c) = 1$? Explain your reasoning.
4. Suppose that $\text{GCD}(a, b) = \text{GCD}(b, c) = 2$. Does that always imply that $\text{GCD}(a, b, c) = 2$? Justify your answer.
5. Is it true that every common divisor of two nonzero whole numbers a and b is a divisor of the $\text{GCD}(a, b)$? Explain your answer.
6. How can you tell from the prime factorization of two numbers if their LCM equals the product of the numbers? Explain your reasoning.
7. Can the LCM of two nonzero whole numbers ever be greater than the product of the two numbers? Explain your reasoning.

Open-Ended

8. Find three pairs of numbers for which the LCM of the numbers in a pair is less than the product of the two numbers.
9. Describe infinitely pairs of numbers whose GCD is equal to the following numbers.
   a. 2 **b.** 6 **c.** 91
10. A large gear is used to turn a smaller gear. If the larger gear makes 72 revolutions per minute and the smaller gear makes 1500 revolutions per minute, how many teeth does each gear have? Give three different possibilities. What is the least number of teeth possible?

Cooperative Learning

11. **a.** In your group, discuss whether the Euclidean algorithm for finding the GCD of two numbers should be introduced in middle school (To all students? To some?). Why or why not?
    b. If you decide that it should be introduced in middle school, discuss how it should be introduced. Report your group's decision to the class.

Connecting Mathematics to the Classroom

12. Describe to a sixth-grade student the difference between a divisor and multiple.
13. Eleanor claims that the $\text{GCD}(0, a) = 0$. Is she correct? What does she understand about GCD? What does she not understand?
14. Aiko says to find the LCM you can just multiply the two numbers. As a teacher, how do you respond?
15. A student wants to know how many whole numbers between 1 and 10,000 inclusive are either multiples of 3 or multiples of 5. She wonders if it is correct to find the number of those whole numbers that are multiples of 3 and add the number of those that are multiples of 5. How do you respond?

Review Problems

16. Find the greatest digit that makes the following statements true.
    a. $3 \mid 83\_51$ **b.** $11 \mid 8\_691$ **c.** $23 \mid 103\_6$
17. Find the prime factorization of the following numbers.
    a. 17,496 **b.** 32,715 **c.** $2^4 \cdot 8^2 \cdot 27^3$
18. Is 2223 prime? Justify your answer.
19. Find a number that has exactly five prime factors.
20. Find the least positive number that is divisible by 2, 4, 6, 8, and 10.
21. What is the greatest prime that must be used to determine if 3359 is prime?

National Assessments

National Assessment of Educational Progress (NAEP) Question

The least common multiple of 8, 12, and a third number is 120. Which of the following could be the third number?

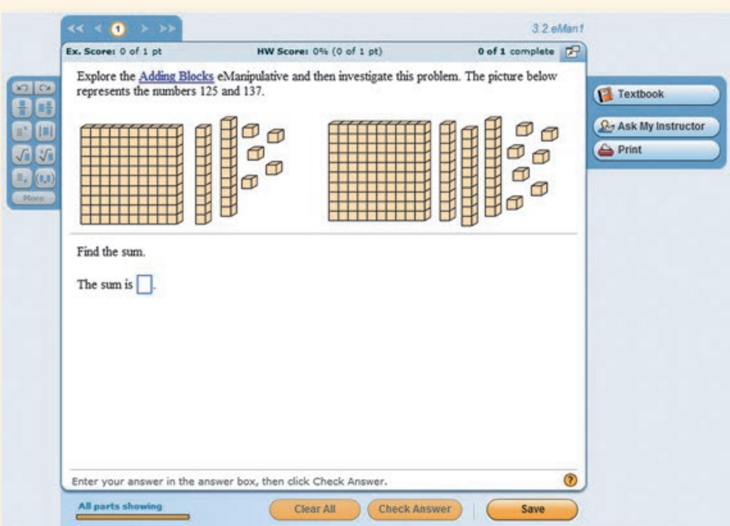
A. 15
B. 16
C. 24
D. 32
E. 48

NAEP, Grade 8, 1990

The Best Online Resource — MyMathLab®

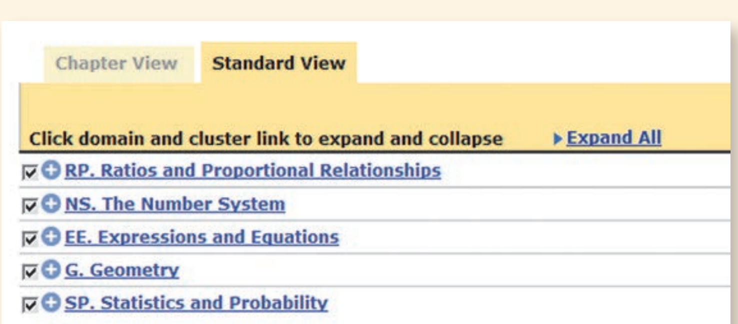
VIDEOS

- **NEW!** Common Core in Action – expert faculty shed light on what the standards really mean for classroom teachers
- **UPDATED!** Section Lectures – address the math topics in each section. Students can navigate directly to specific examples within the video.
- Integrating Mathematics and Pedagogy (IMAP) – feature elementary school children working problems
- Classroom Videos - experienced teachers teaching key topics, with commentary by the teacher and college faculty
- Responding to Student Work – analysis and helpful responses to elementary school children's work



ASSIGNABLE EXERCISES

- Textbook exercises - over 2000 algorithmically generated exercises that parallel those in the text
- E-manipulatives exercises – require use of electronic manipulatives
- Integrating Mathematics and Pedagogy (IMAP) video exercises – require analysis of student work
- **NEW!** Common Core Assessment Analysis exercises – require analysis and interpretation of sample CCSS exercises
- Assessment exercises – hundreds of exercises from the test bank
- Review exercises – for prerequisite math and geometry skills. Use with MyMathLab's personalized homework functionality to target gaps in prerequisite skills.



VIEW BY STANDARDS

You can toggle between viewing course content organized around the Table of Contents or around other standards, including the Common Core Standards.

Chapter 1

An Introduction to Problem Solving

Preliminary Problem

Jill received 10 boxes of coins, each box containing 10 identical looking coins. She knows that one box has 10 counterfeit coins, while all the other coins are genuine. She also knows that each fake coin weighs 1 ounce, while a real coin weighs 2 ounces. Jill has a scale and claims it is possible to determine which is the box with fake coins, in one weighing, as follows:

> "Number the boxes 1 through 10, and take 1 coin from the first box, 2 from the second, 3 from the third, and so on until 10 are taken from the last box. Next, I weigh all the coins taken out, and I can determine which box has the fake coins."

Explain why Jill's scheme would work.

If needed, see Hint on page 35.

IMAP Video

Watch Elsie talk about problem-solving with her students.

Problem solving has long been central in the learning of mathematics at all levels. George Pólya (1887–1985), a great mathematician of the twentieth century, is the father of mathematical problem solving. He pointed out that "solving a problem means finding a way out of difficulty, a way around an obstacle, attaining an aim which was not immediately attainable." (Pólya 1981, p. ix)

Polya developed a four-step problem solving process which has been adopted by many. A modified version is given here.

1. Understanding the problem
2. Devising a plan
3. Carrying out the plan
4. Looking back

CCSS

The *Common Core State Standards for Mathematics* (hereafter referred to as *Common Core Standards* and abbreviated as CCSS) were developed in 2010 through the work of the National Governors Association and the Council of Chief State School Officers. The *Common Core Standards* are built around its *Standards for Mathematical Practice* seen in Table 1.

Table 1

1. Make sense of problems and persevere in solving them. Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping in to a solution attempt.
2. Reason abstractly and quantitatively. Mathematically proficient students make sense of quantities and their relationships in problem situations.
3. Construct viable arguments and critique the reasoning of others. Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples.
4. Model with mathematics. Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community.
5. Use appropriate tools strategically. Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software.
6. Attend to precision. Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem.
7. Look for and make use of structure. Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$.
8. Look for and express regularity in repeated reasoning. Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal.

The *Standards for Mathematical Practice* are used in this book to enhance Polya's four-step process as seen in Table 2.

Table 2

Expanded Four-Step Problem Solving Process with Input from *Standards for Mathematical Practice*

1. **Understanding the problem.**
   - Start by explaining the personal meaning of a problem.
   - Ask if the problem can be stated differently.
   - Analyze goals to identify what is to be found and what is needed.
   - Analyze the givens.
   - Analyze the constraints.
   - Ask what information is missing from the problem.
   - Ask about missing or unneeded information in the problem.
   - Make sense of quantities and their relationships in the problem situation.
   - Look for discernable patterns or structures.
2. **Devising a plan.**
   - Look for a pattern or a structure.
   - Examine related or analogous problems and determine whether the same techniques applied to them can be applied to the current problem.
   - Examine a simpler or special case of the problem to gain insight into the solution of the original problem.
   - Make a table or list.
   - Identify a subgoal.
   - Make a diagram.
   - Use guess and check.
   - Work backward.
   - Write an equation.
   - Abstract a given situation and represent it symbolically.
   - Plan a solution pathway.
   - Make assumptions and approximations to simplify a complicated situation.
   - Use clear definitions.
3. **Carrying out the plan.**
   - State the meaning of any symbols used.
   - Manipulate the representing symbols as if they have a life of their own.
   - Implement the strategy or strategies in step 2 and perform any necessary actions or computations.
   - Attend to the precision in language and mathematics used.
   - Apply the mathematics to solve problems.
   - Check each step of the plan along the way—this may be intuitive checking or formal proof of each step.
   - Keep an accurate record of all work.
   - Map relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas.
   - Use appropriate tools strategically.
   - Look for general methods and for shortcuts to calculations.
   - Detect possible errors using estimation and other mathematical knowledge.
   - Specify units of measure.
4. **Looking back.**
   - Check the results in the original problem (in some cases this will require a proof).
   - Interpret the solution in terms of the original problem: Does the answer make sense? Does it answer the question that was asked?
   - Determine whether there is another method of finding the solution.
   - Improve the model if it has not served its purpose.
   - Maintain oversight of the process.
   - Evaluate the reasonableness of intermediate results.
   - Check answers with a different method.
   - Continually ask: "Does this make sense?"
   - Understand different approaches.
   - Identify correspondences among different approaches.
   - Justify conclusions.
   - Communicate conclusions to others.
   - Respond to arguments of others.
   - If possible, determine other related or more general problems for which the technique will work.

Students learn mathematics as a result of solving problems. *Exercises* are routine practice for skill building and serve a purpose in learning mathematics, but problem solving must be a focus of school mathematics. A reasonable amount of tension and discomfort improves problem-solving performance.

Mathematical problem solving may occur when:

1. Students are presented with a situation that they understand but do not know how to proceed directly to a solution.
2. Students are interested in finding the solution and attempt to do so.
3. Students are required to use mathematical ideas to solve the problem.

We present many opportunities in this text to solve problems. Each chapter opens with a problem that can be solved using concepts developed in the chapter. We give a hint for the solution to the problem at the end of each chapter. Throughout the text, some problems are solved using a four-step process.

Working with others to solve problems enhances problem-solving ability and communication skills. We encourage *cooperative learning* and working in groups whenever possible. To encourage group work and help identify when cooperative learning could be useful, we identify activities and problems where group discussions are especially beneficial for learning mathematics.

1-1 Mathematics and Problem Solving

1-1 Objectives

Students will be able to understand and explain

- The four-step problem-solving process.
- How to solve problems using various problem-solving strategies.

If problems are approached in only one way, a mind-set may be formed. For example, consider the following:

> *Spell the word* spot *three times out loud. "S-P-O-T! S-P-O-T! S-P-O-T!" Now answer the question "What do we do when we come to a green light?" Write an answer.*

If we answer "Stop," we may be guilty of having formed a mind-set. We do not stop at a *green* light.

Consider the following problem: "A shepherd had 36 sheep. All but 10 died. How many lived?" If we answer "10," we are ready to try some problems. If not, we probably did not understand the question by not reading it carefully. *Understanding the problem* is the first step in the four-step problem-solving process.

Strategies for Problem Solving

IMAP Video

Watch Tonya's class model strategies for solving a problem.

We next provide a variety of problems with different contexts to provide experience in problem solving. Strategies are used to discover or construct the means to achieve a solution. For each strategy described, we give an example that can be solved with that strategy. Often, problems can be solved in more than one way. There is no one best strategy to use.

In many of the examples, we use the set of **natural numbers**, 1, 2, 3, ... Note that the first three dots, an *ellipsis*, are used to represent missing terms. The expanded problem-solving steps highlighting some strategies are shown next.

Historical Note

George Pólya (1887–1985) was born in Hungary, moved to the United States in 1940, and after a brief stay at Brown University, joined the faculty at Stanford University. A preeminent mathematician, he also focused on mathematics education. He published 10 books, including *How To Solve It* (1945), which has been translated into 23 languages.

Strategy: Look for a Pattern

IMAP Video

Watch a fourth grade class model Gauss's strategy.

Problem Solving Gauss's Problem

As a student, Carl Gauss and his fellow classmates were asked to find the sum of the first 100 natural numbers. The teacher expected to keep the class occupied for some time, but Gauss gave the answer almost immediately. How might he have done it?

Understanding the Problem The natural numbers are 1, 2, 3, 4, ... Thus, the problem is to find the sum $1 + 2 + 3 + 4 + \ldots + 100$.

Devising a Plan The strategy *look for a pattern* is useful here. One story about young Gauss reports that he listed the sum, and wrote the same sum backwards as in Figure 1. If $S = 1 + 2 + 3 + 4 + 5 + \ldots + 98 + 99 + 100$, then Gauss could have seen the following pattern.

$$\begin{array}{rcccccccc} S = & 1 + & 2 + & 3 + & 4 + & 5 + \ldots + & 98 + & 99 + & 100 \\ + \underline{S} = & \underline{100 +} & \underline{99 +} & \underline{98 +} & \underline{97 +} & \underline{96 + \ldots +} & \underline{3 +} & \underline{2 +} & \underline{1} \\ 2S = & 101 + & 101 + & 101 + & 101 + & 101 + \ldots + & 101 + & 101 + & 101 \end{array}$$

Figure 1

To discover the original sum from the last equation, Gauss could have divided the sum, $2S$, in Figure 1 by 2.

Carrying Out the Plan There are 100 sums of 101. Thus, $2S = 100 \cdot 101$ and $S = \dfrac{100 \cdot 101}{2} = 5050$.

Looking Back Note that the sum in each pair $(1, 100), (2, 99), (3, 98), \ldots, (100, 1)$ is always 101, and there are 100 pairs with this sum. This technique can be used to solve a more general problem of finding the sum of the first n natural numbers $1 + 2 + 3 + 4 + 5 + 6 + \ldots + n$. We use the same plan as before and notice the relationship in Figure 2. Because there are n sums of $n + 1$ we have $2S = n(n + 1)$ and $S = \dfrac{n(n + 1)}{2}$.

$$\begin{array}{rcccccccccc} S = & 1 & + & 2 & + & 3 & + & 4 & + \ldots + & n \\ + \underline{S} = & \underline{n} & \underline{+} & \underline{(n-1)} & \underline{+} & \underline{(n-2)} & \underline{+} & \underline{(n-3)} & \underline{+ \ldots +} & \underline{1} \\ 2S = & (n+1) & + & (n+1) & + & (n+1) & + & (n+1) & + \ldots + & (n+1) \end{array}$$

Figure 2

A different strategy for finding a sum of consecutive natural numbers involves the strategy of *making a diagram* and thinking of the sum geometrically as a stack of blocks. This alternative method is explored in exercise 2 of Assessment 1-1A.

NOW TRY THIS 1

Explain whether the approach in Gauss's Problem of writing the sum backwards and applying the strategy "Look for a Pattern" will or will not work in finding the following sum: $1^2 + 2^2 + \ldots + 100^2$.

Historical Note

Carl Gauss (1777–1855), one of the greatest mathematicians of all time, was born to humble parents in Brunswick, Germany. He was an infant prodigy who later made contributions in many areas of science as well as mathematics. After Gauss's death, the King of Hanover honored him with a commemorative medal with the inscription "Prince of Mathematics."

Strategy: Examine a Related Problem

Problem Solving Sums of Even Natural Numbers

Find the sum of the even natural numbers less than or equal to 100. Generalize the result.

Understanding the Problem Even natural numbers are 2, 4, 6, 8, 10, ... The problem is to find the sum of these numbers: $2 + 4 + 6 + 8 + \ldots + 100$.

Devising a Plan Recognizing that the sum can be *related to Gauss's original problem* helps us devise a plan. Consider the following:

$$\begin{aligned} 2 + 4 + 6 + 8 + \ldots + 100 &= 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + 2 \cdot 4 + \ldots + 2 \cdot 50 \\ &= 2(1 + 2 + 3 + 4 + \ldots + 50) \end{aligned}$$

Thus, we can use Gauss's method to find the sum of the first 50 natural numbers and then double that result.

Carrying Out the Plan We carry out the plan as follows:

$$\begin{aligned} 2 + 4 + 6 + 8 + \ldots + 100 &= 2(1 + 2 + 3 + 4 + \ldots + 50) \\ &= 2\left[\frac{50(50 + 1)}{2}\right] \\ &= 2550 \end{aligned}$$

Thus, the sum of the even natural numbers less than or equal to 100 is 2550.

Looking Back A different way to approach this problem is to realize that there are 25 sums of 102, as shown in Figure 3. (Why are there 25 sums to consider, and why is the sum in each pair always 102?)

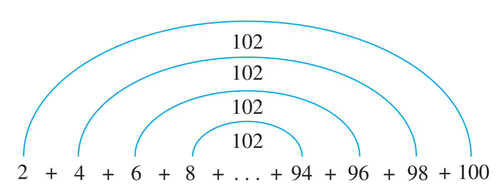


Figure 3

Thus, the sum is $25 \cdot 102 = 2550$.

The numbers 2, 4, 6, 8, . . . , 100 are an example of an *arithmetic sequence*—an ordered list of numbers, or *terms*, in which each term starting from the second one differs from the previous term by the same amount—the common difference. The common difference in the above sequence is 2.

NOW TRY THIS 2

Find the sum of consecutive natural numbers shown: $25 + 26 + 27 + \ldots + 120$. Solve this problem in two different ways.

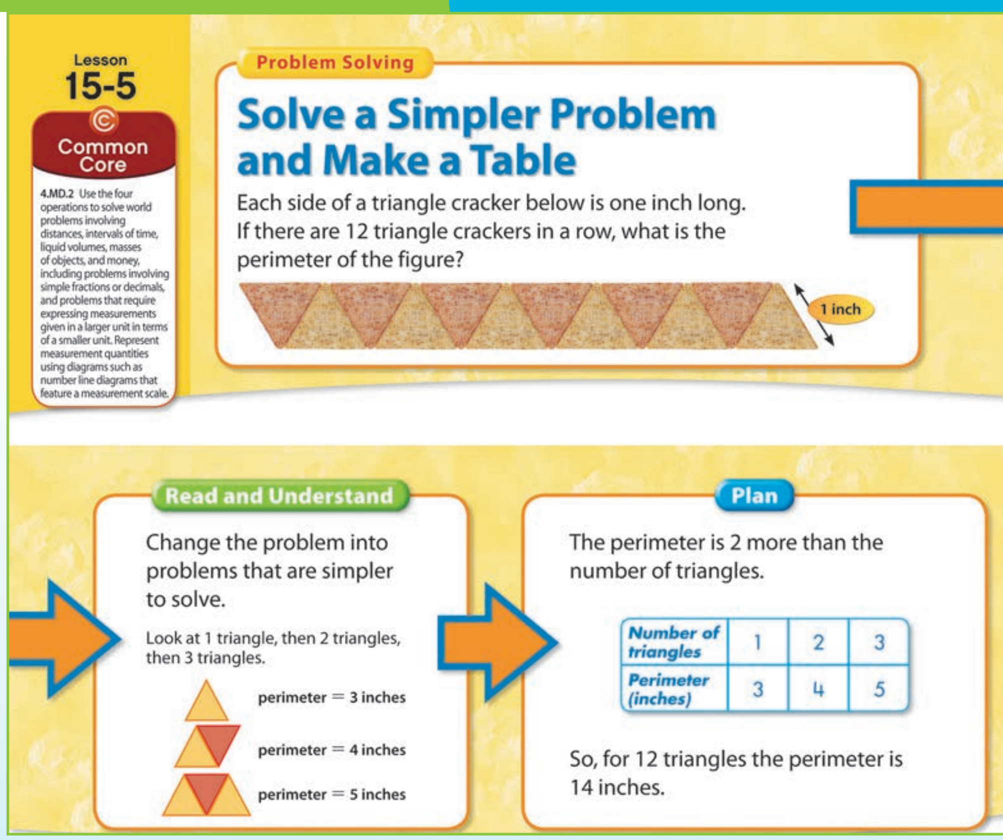
NOW TRY THIS 3

Each of 16 people in a round-robin handball tournament played each other person exactly once. How many games were played?

Strategies: Examine a Simpler Case; Make a Table

Often used strategies in problem solving are *examine a simpler case* and *make a table*. A table can be used to look for patterns that emerge in the problem, which in turn can lead to a solution. An example of these strategies is shown on the grade 4 student page below.

School Book Page Examine a Simpler Case


Lesson 15-5

Common Core

4.MD.2 Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.

Problem Solving

Solve a Simpler Problem and Make a Table

Each side of a triangle cracker below is one inch long. If there are 12 triangle crackers in a row, what is the perimeter of the figure?

1 inch

Read and Understand

Change the problem into problems that are simpler to solve.

Look at 1 triangle, then 2 triangles, then 3 triangles.

perimeter = 3 inches

perimeter = 4 inches

perimeter = 5 inches

Plan

The perimeter is 2 more than the number of triangles.

Number of triangles	1	2	3
Perimeter (inches)	3	4	5

So, for 12 triangles the perimeter is 14 inches.

Strategy: Identify a Subgoal

In attempting to devise a plan for solving a problem, a solution to a somewhat easier or more familiar related problem could make it easier. In such a case, finding the solution to the easier problem may become a *subgoal*. The magic square problem on page 8 shows an example of this.

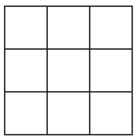


Figure 4

Problem Solving A Magic Square

Arrange the numbers 1 through 9 into a square subdivided into nine smaller squares like the one shown in Figure 4 so that the sum of every row, column, and major diagonal is the same. The result is a *magic square*.

Understanding the Problem Each of the nine numbers 1, 2, 3, . . . , 9 must be placed in the small squares, a different number in each square, so that the sums of the numbers in each row, in each column, and in each of the two major diagonals are the same.

Devising a Plan If we knew the fixed sum of the numbers in each row, column, and diagonal, we would have a better idea of which numbers can appear together in a single row, column, or diagonal. Thus *the subgoal* is to find that fixed sum. The sum of the nine numbers, $1 + 2 + 3 + \ldots + 9$, equals 3 times the sum in one row. (Why?) Consequently, the fixed sum can be found using the process developed by Gauss. We have $\frac{1 + 2 + 3 + \ldots + 9}{3} = \frac{(9 \cdot 10) \div 2}{3} = 15$, so the sum in each row, column, and diagonal must be 15. Next, we need to decide what numbers could occupy the various squares. The number in the center space will appear in four sums, each adding to 15 (two diagonals, the second row, and the second column). Each number in the corners will appear in three sums of 15. (Why?) If we write 15 as a sum of three different numbers 1 through 9 in all possible ways, we could then count how many sums contain each of the numbers 1 through 9. The numbers that appear in at least four sums are candidates for placement in the center square, whereas the numbers that appear in at least three sums are candidates for the corner squares. Thus the new *subgoal* is to write 15 in as many ways as possible as a sum of three different numbers from 1, 2, 3, . . . , 9.

Carrying Out the Plan The sums of 15 can be written systematically as follows:

$$9 + 5 + 1$$
$$9 + 4 + 2$$
$$8 + 6 + 1$$
$$8 + 5 + 2$$
$$8 + 4 + 3$$
$$7 + 6 + 2$$
$$7 + 5 + 3$$
$$6 + 5 + 4$$

Note that the order of the numbers in sums like $9 + 5 + 1$ is irrelevant because the order in which additions are done does not matter. In the list, 1 appears in only two sums, 2 in three sums, 3 in two sums, and so on. Table 3 summarizes this information.

Table 3

Number	1	2	3	4	5	6	7	8	9
Number of sums containing the number	2	3	2	3	4	3	2	3	2

The only number that appears in four sums is 5; hence, 5 must be in the center of the square. (Why?) Because 2, 4, 6, and 8 appear 3 times each, they must go in the corners. Suppose we choose 2 for the upper left corner. Then 8 must be in the lower right corner. This is shown in Figure 5(a). Now we could place 6 in the lower left corner or upper right corner. If we choose the upper right corner, we obtain the result in Figure 5(b). The magic square can now be completed, as shown in Figure 5(c).

2		
	5	
		8

(a)

2		6
	5	
		8

(b)

2	7	6
9	5	1
4	3	8

(c)

Figure 5

Looking Back We have seen that 5 was the only number among the given numbers that could appear in the center. However, we had various choices for a corner, and so it seems that the magic square we found is not the only one possible. Can you find all the others?

Another way to see that 5 could be in the center square is to consider the sums $1 + 9, 2 + 8, 3 + 7, 4 + 6$, as shown in Figure 6. We could add 5 to each to obtain 15.

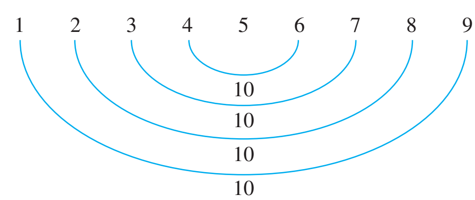


Figure 6

Strategy: Make a Diagram

In the following problem, *making a diagram* helps us to understand the problem and work toward a solution.

Problem Solving 50-m Race Problem

Bill and Jim ran a 50-m race three times. The speed of the runners did not vary. In the first race, Jim was at the 45-m mark when Bill crossed the finish line.

a. In the second race, Jim started 5 m ahead of Bill, who lined up at the starting line. Who won?
b. In the third race, Jim started at the starting line and Bill started 5 m behind. Who won?

Understanding the Problem When Bill and Jim ran a 50-m race, Bill won by 5 m; that is, whenever Bill covered 50 m, at the same time Jim covered only 45 m. If Bill started at the starting line and Jim started at the 5-m line or if Jim started at the starting line and Bill started 5 m behind, we are to determine who would win in each case.

Devising a Plan A strategy to determine the winner under each condition is to *make a diagram*. A diagram for the first 50-m race is given in Figure 7(a). In this case, Bill won by 5 m. In the second race, Jim had a 5-m head start and hence when Bill ran 50 m to the finish line, Jim ran only 45 m. Because Jim is 45 m from the finish line, he reached the finish line at the same time as Bill did. This is shown in Figure 7(b). In the third race, because Bill started 5 m behind, we use Figure 7(a) but move Bill back 5 m, as shown in Figure 7(c). From the diagram we determine the results in each case.